

A GAP THEOREM OF FOUR-DIMENSIONAL GRADIENT SHRINKING SOLITONS

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ABSTRACT. In this paper, we will prove a gap theorem for four-dimensional gradient shrinking soliton. More precisely, we will show that any complete four-dimensional gradient shrinking soliton with nonnegative and bounded Ricci curvature, satisfying a pinched Weyl curvature, either is flat, or $\lambda_1 + \lambda_2 \geq c_0 R > 0$ everywhere for some $c_0 \approx 0.29167$, where $\{\lambda_i\}$ are the two least eigenvalues of Ricci curvature. Furthermore, we will show that $\lambda_1 + \lambda_2 \geq \frac{1}{3}R > 0$ under a better pinched Weyl tensor assumption. We point out that the lower bound $\frac{1}{3}R$ is sharp.

1. INTRODUCTION

A Riemannian manifold (M, g) , couple with a smooth function f , is called gradient Ricci soliton, if there is a constant ρ , such that

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}.$$

The soliton is called shrinking, steady, or expanding, if $\rho > 0$, $\rho = 0$, or $\rho < 0$, respectively. Gradient shrinking solitons (GSS for short) play an important role in the Ricci flow, as they correspond to self-similar solutions, and often arise naturally as limits of dilations of Type I singularities of Ricci flow. They are also generalizations of Einstein metrics. Thus it is a central issue to understand and classify GSS.

The GSS are complete classified in dimension 2 (see [10]) and 3 (see [11, 18, 17, 3]), and in dimension $n \geq 4$ with vanishing Weyl tensor (see [17, 19, 22]). In recent years, there are some other attention to the classification of complete GSS (see [15, 8, 1, 21, 12]).

For a better understanding and ultimately for the classifications of GSS in higher dimension, one tries to obtain some curvature estimates and other geometric structures on GSS. In particular, on a complete non-compact GSS, Chen [7] showed that it will have nonnegative scalar curvature. In addition, Cao-Zhu[5] showed that it has infinite volume (or see [2] Theorem 3.1). While Cao-Zhou[4] obtained a rather precise estimate on asymptotic behavior of the potential function f , and showed that it must have at most Euclidean volume growth.

If the GSS further satisfies some curvature assumptions, then we can get some more precise characteristics. For example, Carrillo-Ni [6] showed that any GSS with nonnegative Ricci curvature must have zero asymptotic volume ratio, and Munteanu-Wang [14] proved that GSS with nonnegative sectional curvature and positive Ricci curvature must be compact. In [13], Munteanu-Wang obtained some

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curvature estimates on four-dimensional GSS with bounded scalar curvature. In this paper, we obtain a gap theorem of four-dimensional GSS with pinched curvature.

Let (M^n, g) be a complete Riemannian manifold. Denote by $Ric = \{R_{ij}\}$ and R are the Ricci tensor and scalar curvature respectively. It is well known that the Riemannian curvature tensor $Rm = \{R_{ijkl}\}$ can be decomposed into the orthogonal components :

$$Rm = W \oplus \frac{2}{n-2} \mathring{Ric} \wedge g \oplus \frac{R}{n(n-1)} g \wedge g,$$

where $W = \{W_{ijkl}\}$ is the Weyl tensor, and $\mathring{Ric} = \{R_{ij} - \frac{R}{n}g_{ij}\}$ is the traceless Ricci curvature. Now we can state our main theorem.

Theorem 1.1. *Let (M^4, g) be a complete four-dimensional GSS with bounded and nonnegative Ricci curvature $0 \leq Ric \leq C$, satisfying*

$$|W| \leq \gamma \left| |\mathring{Ric}| - \frac{1}{2\sqrt{3}}R \right| \quad (*)$$

for some constant $\gamma < 1 + \sqrt{3}$. Then either the soliton is flat, or

$$\lambda_1 + \lambda_2 \geq c_0 R > 0,$$

for some positive constant $c_0 = \frac{(1+2\sqrt{3})-\sqrt{5+4\sqrt{3}}}{2\sqrt{3}} \approx 0.29167$, where λ_1 and λ_2 are the least two eigenvalues of the Ricci curvature.

Remark 1.2. In view of the round cylinder $\mathbb{S}^2 \times \mathbb{R}^2$ with constant scalar curvature, the pinched constant γ in $(*)$ is necessary. Indeed, $\mathbb{S}^2 \times \mathbb{R}^2$ is a non-flat GSS with Ricci curvature $0 \leq Ric \leq \frac{1}{2}R$. Furthermore, $|\mathring{Ric}| = \frac{1}{2}R$, and the Weyl tensor satisfies

$$|W| = \frac{1}{\sqrt{3}}R = (1 + \sqrt{3}) \left| |\mathring{Ric}| - \frac{1}{2\sqrt{3}}R \right|.$$

But the least two eigenvalues of the Ricci curvature $\lambda_1 + \lambda_2 \equiv 0$ everywhere.

Follow by a similar argument, we can show a better result under a better pinched condition as follow.

Theorem 1.3. *Let (M^4, g) be a complete four-dimensional GSS with bounded and nonnegative Ricci curvature $0 \leq Ric \leq C$, satisfying*

$$|W| \leq \gamma \left| |\mathring{Ric}| - \frac{1}{2\sqrt{3}}R \right| \quad (**)$$

for some constant $\gamma \leq \frac{1+\sqrt{3}}{\sqrt{3}}$. Then either the soliton is flat, or

$$\lambda_1 + \lambda_2 \geq \frac{1}{3}R > 0,$$

where λ_1 and λ_2 are the least two eigenvalues of the Ricci curvature.

Remark 1.4. Our conclusion $\lambda_1 + \lambda_2 \geq \frac{1}{3}R > 0$ is sharp due to the example of round cylinder $\mathbb{S}^3 \times \mathbb{R}$. Since $\mathbb{S}^3 \times \mathbb{R}$ is also a non-flat GSS with Ricci curvature $0 \leq Ric \leq \frac{1}{3}R$, and $|\mathring{Ric}| = \frac{1}{2\sqrt{3}}R$, $|W| = 0$. These facts imply that pinched condition $(*)$ holds. But the least two eigenvalues of the Ricci curvature $\lambda_1 + \lambda_2 \equiv \frac{1}{3}R$ everywhere.

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2. PRELIMINARIES

Let (M^4, g_{ij}) be a four-dimensional Riemannian manifold. We can decompose the Riemannian curvature tensor $\{R_{ijkl}\}$ as follows:

$$\begin{aligned} R_{ijkl} = & W_{ijkl} + \frac{1}{2}(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}) \\ & - \frac{1}{6}R(g_{ik}g_{jl} - g_{il}g_{jk}). \end{aligned}$$

In this section, we suppose (M^4, g_{ij}) is a complete Riemannian manifold with bounded curvature. Now we consider the Ricci flow equation

$$\begin{cases} \frac{\partial g_{ij}(x,t)}{\partial t} = -2R_{ij}(x,t), & x \in M^4, t > 0, \\ g_{ij}(x,0) = g_{ij}(x), & x \in M^4. \end{cases}$$

Since the curvature is bounded at the initial metric, it is well known [20] that there exist a complete solution $g(t)$ of the Ricci flow on a time interval $[0, T)$ with bounded curvature for each t . Furthermore, the Ricci curvature tensor $\{R_{ij}\}$ and the scalar curvature R evolve by the (PDE) system under the moving frame (cf. Hamilton [9]):

$$\begin{aligned} \frac{\partial}{\partial t} R_{ij} &= \Delta R_{ij} + 2 \sum_{k,l} R_{ikjl} R_{kl} \\ \frac{\partial}{\partial t} R &= \Delta R + 2|Ric|^2, \end{aligned} \tag{PDE}$$

By using Hamilton's maximum principle for tensor[9], a tensor evolves by a nonlinear heat equation may be controlled by a corresponding (ODE) system. While the (ODE) system corresponding to the above (PDE) is the following

$$\begin{aligned} \frac{d}{dt} R_{ij} &= 2 \sum_{k,l} R_{ikjl} R_{kl} \\ \frac{d}{dt} R &= 2|Ric|^2. \end{aligned} \tag{ODE}$$

By a direct computation, we have the following lemma.

Lemma 2.1. *Let $b = (\lambda_3 + \lambda_4) - (\lambda_1 + \lambda_2)$, where $\{\lambda_i\}$ are the eigenvalues of the Ricci tensor with $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$. Then under the (ODE) system, we have*

$$\frac{1}{2} \frac{d}{dt} b \leq 2b \left(\frac{R}{3} + W_{1212} \right) + (\lambda_1^2 + \lambda_2^2) - (\lambda_3^2 + \lambda_4^2).$$

Proof. Indeed, since $W_{ijij} = W_{klkl}$ and $\sum_i W_{ijij} = 0$ for any orthonormal four-frame $\{e_i, e_j, e_k, e_l\}$, we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} (\lambda_1 + \lambda_2) &\geq \sum_{k=2,3,4} \lambda_k \left(W_{1k1k} + \frac{\lambda_1 + \lambda_k}{2} - \frac{R}{6} \right) \\
&\quad + \sum_{l=1,3,4} \lambda_l \left(W_{2l2l} + \frac{\lambda_2 + \lambda_l}{2} - \frac{R}{6} \right) \\
&= (\lambda_1 + \lambda_2) \left(W_{1212} + \frac{\lambda_1 + \lambda_2}{2} - \frac{R}{6} \right) \\
&\quad + \lambda_3 \left(-W_{1212} + \frac{\lambda_1 + \lambda_2}{2} + \lambda_3 - \frac{R}{3} \right) \\
&\quad + \lambda_4 \left(-W_{1212} + \frac{\lambda_1 + \lambda_2}{2} + \lambda_4 - \frac{R}{3} \right) \\
&= (W_{1212} + \frac{R}{3}) (\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4) + \lambda_3^2 + \lambda_4^2.
\end{aligned}$$

Similarly, we have

$$\frac{1}{2} \frac{d}{dt} (\lambda_3 + \lambda_4) \leq (W_{3434} + \frac{R}{3}) (\lambda_3 + \lambda_4 - \lambda_1 - \lambda_2) + \lambda_1^2 + \lambda_2^2.$$

Subtract the above two inequalities, we obtain our assertion. \square

3. A KEY PINCHED ESTIMATE

In this section, we will give a pinched estimate, which implies that the curvature b described in Lemma 2.1 can become better and better under the Ricci flow.

Lemma 3.1. *Suppose we have a solution of Ricci flow $g(t)_{t \in [0, T]}$ on a four-manifold with uniformly bounded and nonnegative Ricci curvature, and satisfying the pinched condition $(*)$ for all $t \in [0, T]$.*

If $R \geq r_0$ and $b \leq \eta_0 R \leq R$ holds for some positive constant $r_0 > 0$ and $\eta_0 > \tilde{c}$ at time $t = 0$, where $\tilde{c} = \frac{\sqrt{5+4\sqrt{3}}-(1+\sqrt{3})}{\sqrt{3}} \approx 0.41666 > \frac{1}{3}$. Then there exist a positive constant $\delta = \delta(r_0, \eta_0, \gamma) \in (0, 1]$, such that

$$b \leq (\eta_0 - \delta t) R$$

holds for all $t \in [0, T']$, where $T' = \min\{T, \frac{\eta_0 - \tilde{c}}{2}\}$.

Proof. Note that both the Ricci curvature tensor and the Weyl tensor are uniformly bounded, hence $g(t)$ has uniformly bounded curvature.

Consider the set $\Omega(t)_{t \in [0, T']}$ of matrices defined by the inequalities

$$\Omega(t) : \begin{cases} R \geq r_0, \\ b \leq (\eta_0 - \delta t) R. \end{cases}$$

The constant $\delta \in (0, 1]$ will be choosed later.

It is easy to see that $\Omega(t)$ is closed, convex and $O(n)$ -invariant. By the assumptions at $t = 0$ and the Hamilton's maximum principle for tensor, we only need to show the set $\Omega(t)$ is preserved by the (ODE) system. Indeed, we only need to look at points on the boundary of the set.

From the (ODE) system, we have

$$\frac{d}{dt}R = 2|Ric|^2 \geq 0,$$

which implies that $R \geq r_0$ for all $t \geq 0$. Thus the first inequality is preserved. To prove the second inequality, we only need to show that

$$\frac{1}{2}b' \leq (\eta_0 - \delta t)\frac{1}{2}R' - \frac{\delta}{2}R = \eta \cdot \frac{1}{2}R' - \frac{\delta}{2}R,$$

where $b = (\eta_0 - \delta t)R = \eta R$.

By Lemma 2.1 and the (ODE) system, it is suffice to show that

$$2b\left(\frac{R}{3} + W_{1212}\right) + (\lambda_1^2 + \lambda_2^2) - (\lambda_3^2 + \lambda_4^2) \leq \eta \sum_i \lambda_i^2 - \frac{\delta}{2}R.$$

It is equivalent to show that

$$I = (1 + \eta)(\lambda_3^2 + \lambda_4^2) - (1 - \eta)(\lambda_1^2 + \lambda_2^2) - 2\eta R\left(\frac{R}{3} + W_{1212}\right) \geq \frac{\delta}{2}R. \quad (3.1)$$

Now $b = \eta R$, thus $\lambda_3 + \lambda_4 = \frac{1+\eta}{2}R$ and $\lambda_1 + \lambda_2 = \frac{1-\eta}{2}R$. Denote by $x = \frac{\lambda_2 - \lambda_1}{2}$ and $y = \frac{\lambda_4 - \lambda_3}{2}$, which satisfies

$$0 \leq x \leq \frac{1-\eta}{4}R, \quad y \geq 0, \quad x + y \leq \frac{\eta}{2}R.$$

And then

$$\begin{aligned} \lambda_1 &= \frac{1-\eta}{4}R - x, & \lambda_2 &= \frac{1-\eta}{4}R + x, \\ \lambda_3 &= \frac{1+\eta}{4}R - y, & \lambda_4 &= \frac{1+\eta}{4}R + y. \end{aligned}$$

Meanwhile, by a direct computation, we have

$$W_{12}^2 \leq \frac{2}{3} \sum W_{1k}^2 \leq \frac{2}{3} \cdot \frac{1}{8} |W|^2 \leq \frac{1}{12} \gamma^2 \left(|Ric| - \frac{1}{2\sqrt{3}}R \right)^2.$$

In the following, we divide the argument into two cases.

Case 1: $|Ric| \geq \frac{R}{2\sqrt{3}}$. In this case,

$$\begin{aligned} |Ric|^2 &= \sum_i \left(\frac{R}{4} - \lambda_i \right)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 - \frac{1}{4}R^2 \\ &= \left(\frac{1-\eta}{2} \right)^2 R^2 - 2\lambda_1\lambda_2 + 2\left(\frac{1+\eta}{4} \right)^2 R^2 + 2y^2 - \frac{1}{4}R^2 \\ &= \frac{1}{8} \cdot (3\eta^2 - 2\eta + 1)R^2 + 2y^2 - 2\lambda_1\lambda_2. \end{aligned}$$

Thus

$$W_{1212} \leq \frac{\gamma}{2\sqrt{3}} \cdot \left(\sqrt{\frac{1}{8} \cdot (3\eta^2 - 2\eta + 1)R^2 + 2y^2 - 2\lambda_1\lambda_2} - \frac{R}{2\sqrt{3}} \right).$$

So I defined in (3.1) can be calculated as follow :

$$\begin{aligned}
I &= (1+\eta) \left[2 \left(\frac{1+\eta}{4} R \right)^2 + 2y^2 \right] - (1-\eta) \left[\left(\frac{1-\eta}{2} R \right)^2 - 2\lambda_1\lambda_2 \right] \\
&\quad - \frac{2}{3} \eta R^2 - 2\eta R W_{1212} \\
&\geq \frac{1}{24} (-3 + 11\eta - 9\eta^2 + 9\eta^3) R^2 + 2(1+\eta)y^2 + 2(1-\eta)\lambda_1\lambda_2 \\
&\quad - 2\eta R \cdot \frac{\gamma}{2\sqrt{3}} \cdot \left(\sqrt{\frac{1}{8} \cdot (3\eta^2 - 2\eta + 1) R^2 + 2y^2} - 2\lambda_1\lambda_2 - \frac{R}{2\sqrt{3}} \right) \\
&\geq \frac{1}{24} (-3 + 11\eta - 9\eta^2 + 9\eta^3) R^2 + 2(1+\eta)y^2 \\
&\quad - \frac{\gamma\eta R}{\sqrt{3}} \cdot \left[\sqrt{\frac{1}{8} \cdot (3\eta^2 - 2\eta + 1) R^2 + 2y^2} - \frac{R}{2\sqrt{3}} \right].
\end{aligned}$$

Denote by $t = \sqrt{\frac{1}{8} \cdot (3\eta^2 - 2\eta + 1) R^2 + 2y^2}$. Since $y \geq 0$, we then have

$$t \geq \frac{R}{2\sqrt{2}} \sqrt{3\eta^2 - 2\eta + 1}.$$

And the above inequality becomes

$$\begin{aligned}
I &\geq \frac{1}{24} (-3 + 11\eta - 9\eta^2 + 9\eta^3) R^2 \\
&\quad + (1+\eta) \left[t^2 - \frac{1}{8} \cdot (3\eta^2 - 2\eta + 1) R^2 \right] - \frac{\gamma\eta R}{\sqrt{3}} \cdot \left(t - \frac{R}{2\sqrt{3}} \right) \\
&= (1+\eta)t^2 - \frac{\gamma\eta R}{\sqrt{3}} \cdot t \\
&\quad + \frac{1}{24} (-3 + 11\eta - 9\eta^2 + 9\eta^3) R^2 - \frac{1}{8} \cdot (1+\eta) \cdot (3\eta^2 - 2\eta + 1) R^2 + \frac{\gamma\eta}{6} R^2
\end{aligned}$$

The RHS is a quadratic function of t , and increase respect to t . Indeed, we only need to show that $2(1+\eta)t - \frac{\gamma\eta R}{\sqrt{3}} > 0$ holds for all $t \geq \frac{R}{2\sqrt{2}} \sqrt{3\eta^2 - 2\eta + 1}$. It is easy to see that $12(\sqrt{3}-1) > (1+\sqrt{3})^2 > \gamma^2$, and then we have

$$\frac{(1+\eta)\sqrt{3\eta^2 - 2\eta + 1}}{\eta} = \left(\frac{1}{\sqrt{\eta}} + \sqrt{\eta} \right) \sqrt{3\eta - 2 + \frac{1}{\eta}} \geq 2\sqrt{2\sqrt{3}-2} > \gamma.$$

Thus $2(1+\eta)t \geq \frac{R}{\sqrt{2}} \cdot (1+\eta) \sqrt{3\eta^2 - 2\eta + 1} > \frac{R}{\sqrt{2}} \cdot \eta\gamma > \frac{\gamma\eta R}{\sqrt{3}}$.

The above monotonic property respect to t implies that the RHS achieves the minimum value if and only if t takes the minimum value, which is equivalent with

$y = 0$. Thus

$$\begin{aligned}
I &\geq \frac{1}{24}(-3 + 11\eta - 9\eta^2 + 9\eta^3)R^2 - \frac{\gamma\eta R^2}{\sqrt{3}} \cdot \left(\frac{1}{2\sqrt{2}}\sqrt{3\eta^2 - 2\eta + 1} - \frac{1}{2\sqrt{3}} \right) \\
&= \frac{R^2}{24} \left[9\left(\eta - \frac{1}{3}\right) \left(\left(\eta - \frac{1}{3}\right)^2 + \frac{8}{9} \right) - 4\gamma\eta \cdot \left(\sqrt{1 + \frac{9}{2}\left(\eta - \frac{1}{3}\right)^2} - 1 \right) \right] \\
&= \frac{3(\eta - \frac{1}{3})R^2}{8} \left[\left(\eta - \frac{1}{3}\right)^2 + \frac{8}{9} - 2\gamma \cdot \frac{\eta(\eta - \frac{1}{3})}{\sqrt{1 + \frac{9}{2}\left(\eta - \frac{1}{3}\right)^2} + 1} \right] \\
&= \frac{3(\eta - \frac{1}{3})R^2}{8} \left[II + 2(1 + \sqrt{3} - \gamma) \cdot \frac{\eta(\eta - \frac{1}{3})}{\sqrt{1 + \frac{9}{2}\left(\eta - \frac{1}{3}\right)^2} + 1} \right],
\end{aligned}$$

where

$$\begin{aligned}
II &= \left(\eta - \frac{1}{3}\right)^2 + \frac{8}{9} - 2(1 + \sqrt{3}) \cdot \frac{\eta(\eta - \frac{1}{3})}{\sqrt{1 + \frac{9}{2}\left(\eta - \frac{1}{3}\right)^2} + 1} \\
&= \eta^2 - \frac{2}{3}\eta + 1 - 2\eta\left(\eta - \frac{1}{3}\right) \\
&\quad - 2\eta\left(\eta - \frac{1}{3}\right) \cdot \left[\frac{1 + \sqrt{3}}{\sqrt{1 + \frac{9}{2}\left(\eta - \frac{1}{3}\right)^2} + 1} - 1 \right] \\
&= (1 - \eta)(1 + \eta) - 2\eta\left(\eta - \frac{1}{3}\right) \cdot \frac{\sqrt{3} - \sqrt{1 + \frac{9}{2}\left(\eta - \frac{1}{3}\right)^2}}{\sqrt{1 + \frac{9}{2}\left(\eta - \frac{1}{3}\right)^2} + 1} \\
&= (1 - \eta)(1 + \eta) - 2\eta\left(\eta - \frac{1}{3}\right) \cdot \frac{1}{\sqrt{1 + \frac{9}{2}\left(\eta - \frac{1}{3}\right)^2} + 1} \cdot \frac{\frac{3}{2}(1 - \eta)(1 + 3\eta)}{\sqrt{3} + \sqrt{1 + \frac{9}{2}\left(\eta - \frac{1}{3}\right)^2}}.
\end{aligned}$$

Note that $\eta = \eta_0 - \delta t \in [\frac{\eta_0 + \bar{c}}{2}, 1] \subset (\frac{1}{3}, 1]$, thus

$$\begin{aligned}
II &\geq (1 + \eta)(1 - \eta) - 2\eta \cdot \frac{2}{3} \cdot \frac{1}{1 + 1} \cdot \frac{\frac{3}{2}(1 - \eta) \cdot 4}{\sqrt{3} + 1} \\
&= (1 - \eta) \left(1 + \eta - \frac{4}{1 + \sqrt{3}}\eta \right) \geq 0,
\end{aligned}$$

and then

$$\begin{aligned}
I &\geq \frac{3(\eta - \frac{1}{3})R^2}{8} \cdot 2(1 + \sqrt{3} - \gamma) \cdot \frac{\eta(\eta - \frac{1}{3})}{\sqrt{3} + 1} \\
&\geq C_1(\eta_0, \gamma)R^2 \geq C_2(c_0, \eta_0, \gamma)R
\end{aligned}$$

for some positive constant $C_2(r_0, \eta_0, \gamma) > 0$.

Case 2: $|\mathring{Ric}| < \frac{R}{2\sqrt{3}}$. In this case,

$$|\mathring{Ric}|^2 = 2\left(\frac{1 - \eta}{4}\right)^2 + 2\left(\frac{1 - \eta}{4}\right)^2 - \frac{1}{4} + 2y^2 + 2x^2 = \frac{1}{4}\eta^2 R^2 + 2y^2 + 2x^2,$$

which implies that $\eta = \eta_0 - \delta t \in [\frac{\eta_0 + \tilde{c}}{2}, \frac{1}{\sqrt{3}}) \subset (\tilde{c}, \frac{1}{\sqrt{3}}) \subset (\frac{1}{3}, \frac{1}{\sqrt{3}})$, and

$$-W_{1212} \geq \frac{\gamma}{2\sqrt{3}} \cdot \left(\sqrt{\frac{1}{4}\eta^2 R^2 + 2y^2 + 2x^2} - \frac{R}{2\sqrt{3}} \right).$$

By a direct computation, we have

$$\begin{aligned} I &\geq \frac{(1+\eta)^3}{8}R^2 - \frac{(1-\eta)^3}{8}R^2 - \frac{2}{3}\eta R^2 + 2(1+\eta)y^2 - 2(1-\eta)x^2 \\ &\quad + 2\eta R \cdot \frac{\gamma}{2\sqrt{3}} \cdot \left(\sqrt{\frac{1}{4}\eta^2 R^2 + 2y^2 + 2x^2} - \frac{R}{2\sqrt{3}} \right) \\ &= \frac{\eta}{12}(3\eta^2 + 1)R^2 + 2(1+\eta)y^2 - 2(1-\eta)x^2 \\ &\quad + \frac{\gamma\eta R}{\sqrt{3}} \cdot \left(\sqrt{\frac{1}{4}\eta^2 R^2 + 2y^2 + 2x^2} - \frac{R}{2\sqrt{3}} \right) \\ &\geq \frac{\eta}{12}(3\eta^2 + 1)R^2 - 2(1-\eta)x^2 + \frac{\gamma\eta R}{\sqrt{3}} \cdot \left(\sqrt{\frac{1}{4}\eta^2 R^2 + 2x^2} - \frac{R}{2\sqrt{3}} \right). \end{aligned}$$

Denote by $\tau = \sqrt{\frac{1}{4}\eta^2 R^2 + 2x^2}$. Since $x \in [0, \frac{1-\eta}{4}R]$, we then have

$$\tau \in \left[\frac{\eta}{2}R, \frac{R}{2\sqrt{2}}\sqrt{3\eta^2 - 2\eta + 1} \right], \text{ and}$$

$$I \geq \frac{\eta}{12}(3\eta^2 + 1)R^2 - (1-\eta)(\tau^2 - \frac{1}{4}\eta^2 R^2) + \frac{\gamma\eta R}{\sqrt{3}} \cdot \left(\tau - \frac{R}{2\sqrt{3}} \right).$$

Similarly, the RHS is a quadratic function of τ , and it can achieve the minimum value where t take endpoint values, which is equivalent with $x = 0$ or $x = \frac{1-\eta}{4}R$.

If $x = \frac{1-\eta}{4}R$. Then

$$\begin{aligned} RHS &\geq \frac{\eta R^2}{12}(3\eta^2 + 1) - \frac{1}{8}(1-\eta)^3 R^2 \\ &\quad + \frac{\gamma\eta R^2}{\sqrt{3}} \cdot \left(\frac{1}{2\sqrt{2}}\sqrt{3\eta^2 - 2\eta + 1} - \frac{1}{2\sqrt{3}} \right). \end{aligned}$$

Since $\eta > \frac{1}{3}$, we have $\frac{1}{2\sqrt{2}}\sqrt{3\eta^2 - 2\eta + 1} > \frac{1}{2\sqrt{2}}\sqrt{\frac{2}{3}} = \frac{1}{2\sqrt{3}}$. Thus

$$\begin{aligned} RHS &> \frac{\eta R^2}{12}(3\eta^2 + 1) - \frac{1}{8}(1-\eta)^3 R^2 \\ &= \frac{R^2}{24} [2\eta \cdot (3\eta^2 + 1) - 3(1-\eta)^3] \\ &= \frac{R^2}{24} (9\eta^3 - 9\eta^2 + 11\eta - 3) \\ &= \frac{R^2}{24} \left(\eta - \frac{1}{3} \right) [(3\eta - 1)^2 + 8] = C_3(\eta_0, c_0)R. \end{aligned}$$

If $x = 0$. Then

$$\begin{aligned} RHS &\geq \frac{\eta}{12}(3\eta^2 + 1)R^2 + \frac{\gamma\eta R^2}{\sqrt{3}} \cdot \left(\frac{\eta}{2} - \frac{1}{2\sqrt{3}} \right) \\ &= \frac{\eta R^2}{12} [3\eta^2 + 1 + 2\gamma(\sqrt{3}\eta - 1)]. \end{aligned} \tag{3.2}$$

Note that $\sqrt{3}\eta - 1 < 0$, we have

$$\begin{aligned}
& 3\eta^2 + 1 + 2\gamma(\sqrt{3}\eta - 1) \\
& > 3\eta^2 + 1 + 2(1 + \sqrt{3}) \cdot (\sqrt{3}\eta - 1) \\
& = 3\eta^2 + (6 + 2\sqrt{3})\eta - (1 + 2\sqrt{3}) \\
& = 3\left(\eta - \frac{\sqrt{5+4\sqrt{3}} - (1 + \sqrt{3})}{\sqrt{3}}\right) \cdot \left(\eta + \frac{\sqrt{5+4\sqrt{3}} + (1 + \sqrt{3})}{\sqrt{3}}\right).
\end{aligned}$$

Thus

$$RHS \geq C_4(\eta_0, c_0)R.$$

Combine the above argument, we have

$$I \geq C_5(c_0, \eta_0)R.$$

So by choosing $\delta = \delta(c_0, \eta_0, \gamma) = \min\{1, 2C_2, 2C_5\}$, the inequality (3.1) holds. The proof of Lemma 3.1 is complete. \square

4. A GAP THEOREM OF FOUR-DIMENSIONAL SHRINKING GRS

Suppose (M^4, g) is a complete GSS. Then there are a smooth function f and a positive constant ρ , such that

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}.$$

It is well known that there exist a self-similar solution of Ricci flow as follow

$$g(t) = \tau(t)\varphi_t^*(g), \quad t \in (-\infty, \frac{1}{2\rho}),$$

where $\tau(t) = 1 - 2\rho t$, and φ_t is a family of diffeomorphisms.

Now we can prove Theorem 1.1.

Proof. of Theorem 1.1. It is well known that any shrinking GRS with nonnegative Ricci curvature either is flat, or has positive scalar curvature $R \geq r_0 > 0$ for some positive constant $r_0 = r_0(g)$. In the following, we always assume the soliton has positive scalar curvature $R \geq r_0 > 0$ (cf. [16]).

We will argue by contradiction. Denote by

$$\eta_0 = \sup_{x \in (M^4, g)} \frac{b(x)}{R(x)} \leq 1.$$

If $\eta_0 \leq \tilde{c}$, then we have $\lambda_1 + \lambda_2 \geq \frac{1-\tilde{c}}{2}R = c_0R$, and we have done. If not, then $\eta_0 > \tilde{c}$. By the assumptions, we see that the self-similar solution $g(t)_{t \in [0, \frac{1}{10\rho}]}$ has nonnegative and uniformly bounded Ricci curvature with $g(0) = g$.

Then by Lemma 3.1, there exist a positive constant $\delta = \delta(r_0, \eta_0, \gamma) \in (0, 1]$, such that

$$b \leq (\eta_0 - \delta t)R.$$

is preserved under the Ricci flow for all small $t \in [0, T']$, where $T' = \min\{\frac{1}{10\rho}, \frac{\eta_0 - \tilde{c}}{2}\}$.

Hence we have

$$b \leq (\eta_0 - \delta T')R$$

at every point. But this is impossible. Since there exist some point $p \in M$, such that $b(p) \geq (\eta_0 - \frac{\delta}{2}T')R(p)$ at time $t = 0$. Note that $g(t)$ only changes by scaling

and a diffeomorphism on M^4 , and then exist some point $q \in M$, such that at time $t = T'$,

$$b(q, T') = \frac{1}{1 - 2\rho T'} b(p) \geq \frac{1}{1 - 2\rho T'} (\eta_0 - \frac{\delta}{2} T') R(p) = (\eta_0 - \frac{\delta}{2} T') R(q, T'),$$

which is contradictive with $b(q, T') \leq (\eta_0 - \delta T') R(q, T')$.

And we complete the proof of Theorem 1.1. □

Next, we follow a similar argument to prove Theorem 1.3.

Proof. of Theorem 1.3. Obviously, we only need to show that

$$\eta_0 = \sup_{x \in (M^4, g)} \frac{b(x)}{R(x)} \leq \frac{1}{3}.$$

If not, $\eta_0 > \frac{1}{3}$, then we can prove the following assertion.

Claim 4.1. *Suppose we have a solution of Ricci flow $g(t)_{t \in [0, T]}$ on a four-manifold with uniformly bounded and nonnegative Ricci curvature, and satisfying the pinched condition (**) for all $t \in [0, T]$.*

If $R \geq r_0$ and $b \leq \eta_0 R$ holds for some positive constant $r_0 > 0$ and $\eta_0 > \frac{1}{3}$ at time $t = 0$. Then there exist a positive constant $\delta = \delta(r_0, \eta_0, \gamma) \in (0, 1]$, such that

$$b \leq (\eta_0 - \delta t) R$$

holds for all $t \in [0, T']$, where $T' = \min\{T, \frac{\eta_0 - \frac{1}{3}}{\delta}\}$.

For the proof of Claim 4.1, we check the argument of Lemma 3.1. Then we only need to get a positive lower bound of (3.2). Indeed,

$$\begin{aligned} 3\eta^2 + 1 + 2\gamma(\sqrt{3}\eta - 1) &\geq 3\eta^2 + 1 + 2 \cdot \frac{1 + \sqrt{3}}{\sqrt{3}} \cdot (\sqrt{3}\eta - 1) \\ &= 3(\eta - \frac{1}{3}) \cdot (\eta + \frac{2 + \sqrt{3}}{\sqrt{3}}) \geq C(\eta_0). \end{aligned}$$

Thus Claim 4.1 holds, which develops a contradiction. And then we obtain Theorem 1.3. □

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